Minimizer of an isoperimetric ratio on a metric on $\mathbb{R}^2$ with finite total area

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Abstract

Let $g = (g_{ij})$ be a complete Riemmanian metric on $\mathbb{R}^2$ with finite total area and $I_g = \inf_\gamma I(\gamma)$ with $I(\gamma) = L(\gamma)(A_{in}(\gamma)^{-1} + A_{out}(\gamma)^{-1})$ where γ is any closed simple curve in $\mathbb{R}^2$, $L(\gamma)$ is the length of γ, $A_{in}(\gamma)$ and $A_{out}(\gamma)$ are the areas of the regions inside and outside γ respectively, with respect to the metric g. Under some mild growth conditions on g we prove the existence of a minimizer for I_g. As a corollary we obtain a new proof for the existence of a minimizer for $I_{g(t)}$ for any $0 < t < T$ when the metric $g(t) = g_{ij}(\cdot, t) = u\delta_{ij}$ is the maximal solution of the Ricci flow equation $\partial g_{ij}/\partial t = -2R_{ij}$ on $\mathbb{R}^2 \times (0, T)$ [DH] where $T > 0$ is the extinction time of the solution.

Key words: existence of minimizer, isoperimetric ratio, complete Riemannian metric on $\mathbb{R}^2$, finite total area

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Isoperimetric inequalities arises in many problems on analysis and geometry such as the study of partial differential equations and Sobolev inequality [B], [SY], [T1]. Isoperimetric inequalities are also used by N.S. Trudinger [T2] in the study of sharp estimates for the Hessian equations and Hessian integrals. In [G], [H1], M. Gage and R. Hamilton studied isoperimetric inequalities arising from the curve shortening flow. In [DH], [DHS] and [H2], P. Daskalopoulos, R. Hamilton, N. Sesum, studied isoperimetric inequalities in Ricci flow and used it to study the behavior of solutions of Ricci flow which is an important tool in the classification of manifolds [MT], [P1], [P2], [Z].

Let $g = (g_{ij})$ be a complete Riemmanian metric on $\mathbb{R}^2$ with finite total area $A = \int_{\mathbb{R}^2} dV_g$ satisfying

$$\lambda_1(|x|)\delta_{ij} \le g_{ij}(x) \le \lambda_2(|x|)\delta_{ij} \quad \forall |x| \ge r_0 \tag{1}$$

for some constant $r_0 > 1$ and positive monotone decreasing functions $\lambda_1(r)$, $\lambda_2(r)$, on $[r_0, \infty)$ that satisfy

$$\int_r^{c_0 r} \sqrt{\lambda_1(\rho)}\, d\rho \ge \pi r \sqrt{\lambda_2(r)} \quad \forall r \ge r_0, \tag{2}$$

$$r\sqrt{\lambda_1(c_0 r)} \ge b_1 \int_r^{\infty} \rho\lambda_2(\rho)\, d\rho \quad \forall r \ge r_0, \tag{3}$$

$$\int_r^{r^2} \sqrt{\lambda_1(\rho)}\, d\rho \ge b_2 \quad \forall r \ge r_0, \tag{4}$$

and

$$\lambda_1(c_0 r) \ge \delta\lambda_2(r) \quad \forall r \ge r_0 \tag{5}$$

for some constants $c_0 > 1$, $b_1 > 0$, $b_2 > 0$, $\delta > 0$, where $|x|$ is the distance of x from the origin with respect to the Euclidean metric. For any closed simple curve γ in $\mathbb{R}^2$, let (cf. [DH])

$$I(\gamma) = L(\gamma)\left(\frac{1}{A_{in}(\gamma)} + \frac{1}{A_{out}(\gamma)}\right) \tag{6}$$

where $L(\gamma)$ is the length of the curve γ, $A_{in}(\gamma)$ and $A_{out}(\gamma)$ are the areas of the regions inside and outside γ respectively, with respect to the metric g. Let

$$I = I_g = \inf_{\gamma} I(\gamma) \tag{7}$$

where the infimum is over all closed simple curves γ in $\mathbb{R}^2$. In this paper we will prove that there exists a constant $b_0 > 0$ such that if the isoperimetric ratio $I_g < b_0$, then there exists a closed simple curve γ satisfying $I_g = I(\gamma)$. As a corollary we obtain a new proof for the existence of a minimizer for the isoperimetric ratio $I_{g(t)}$ for any $0 < t < T$ when the metric $g(t) = g_{ij}(\cdot, t) = u\delta_{ij}$ is the maximal solution of the Ricci flow [DH]

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \quad \text{ on } \mathbb{R}^2 \times (0, T)$$

where $T > 0$ is the extinction time of the solution and u is a solution of

$$u_t = \Delta \log u \quad \text{ on } \mathbb{R}^2 \times (0, T). \tag{8}$$

We will use an adaptation of the technique of [H1] and [H2] to prove the result. In [H1], [H2], since the domain under consideration is either the sphere S^2 ([H2]) or bounded domain in $\mathbb{R}^2$ ([H1]), the minimizing sequences for the infimum of the

isoperimetric ratios considered in [H1], [H2], stay in a compact set. On the other hand since the isoperimetric ratio (6) is for any curve γ in $\mathbb{R}^2$, the minimizing sequence of curves for the infimum of the isoperimetric ratio (7) may not stay in a compact subset of $\mathbb{R}^2$ and may not have a limit at all. So we will need to show that there exists a constant such that this is impossible when I_g is less than this constant. After this we will use the curve shortening flow technique of [H2] to modify the minimizing sequence of curves and show that they will converge to a minimizer of (7).

For any $x_0 \in \mathbb{R}^2$ and $r > 0$ let $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$ and $B_r = B_r(0)$. The main results of the paper are as follows.

Theorem 1. *Suppose g satisfies (1) for some constant $r_0 > 1$ where $\lambda_1(r)$, $\lambda_2(r)$, are positive monotone decreasing functions on $[r_0, \infty)$ that satisfy (2), (3), (4) and (5) for some constants $c_0 > 1$, $b_1 > 0$, $b_2 > 0$ and $\delta > 0$. Then there exists a constant $b_0 > 0$ depending on b_1, b_2 and A such that the following holds. If*

$$I_g < b_0, \tag{9}$$

then there exists a closed simple curve γ in $\mathbb{R}^2$ such that $I_g = I(\gamma)$. Hence $I_g > 0$.

Proposition 2. *Suppose $g = (g_{ij})$ satisfies*

$$\frac{C_1}{r^2(\log r)^2}\delta_{ij} \le g_{ij} \le \frac{C_2}{r^2(\log r)^2}\delta_{ij} \quad \forall r \ge r_1$$

for some constants $C_2 \ge C_1 > 0$, $r_1 > 1$. Then there exist constants $c_0 > 1$, $\delta > 0$, $b_1 > 0$, $b_2 > 0$, and $r_0 \ge r_1$ such that (2), (3), (4) and (5) hold.

Corollary 3. *Let $g_{ij}(x,t) = u(x,t)\delta_{ij}$ where u is the maximal solution of (8) with initial value $0 \le u_0 \in L^p(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $u_0 \not\equiv 0$, for some $p > 1$ satisfying*

$$u_0(x) \le \frac{C}{|x|^2(\log|x|)^2} \quad \forall |x| > 1 \tag{10}$$

given by [DP] and [Hu] where $T = (1/4\pi)\int_{\mathbb{R}^2} u_0\,dx$. Then for any $0 < t_1 < T$ there exists a constant $b_0 > 0$ such that the following holds. For any $t_1 \le t < T$, if $I_{g(t)} < b_0$, then there exists a closed simple curve γ that satisfies $I_{g(t)} = I(\gamma)$.

Proof of Proposition 2: Let $\lambda_i(r) = C_i(r\log r)^{-2}$, $i = 1, 2$,

$$c_0 = 2e^{\pi\sqrt{C_2/C_1}}, \tag{11}$$

and $\delta = C_1/(2c_0^2 C_2)$. We choose $r_2 \ge r_1$ such that

$$\frac{\log r}{\log(c_0 r)} \ge \frac{1}{\sqrt{2}} \quad \forall r \ge r_2. \tag{12}$$

Then by (11) and (12),

$$\frac{\lambda_1(c_0 r)}{\lambda_2(r)} = \frac{C_1}{c_0^2 C_2}\left(\frac{\log r}{\log(c_0 r)}\right)^2 \ge \frac{C_1}{2c_0^2 C_2} = \delta \quad \forall r \ge r_2. \tag{13}$$

We next note that

$$\lim_{r\to\infty}\left((\log r)\log\left(\frac{\log(c_0 r)}{\log r}\right)\right) = \lim_{z\to 0}\frac{\log((\log c_0)z+1)}{z} = \log c_0. \tag{14}$$

By (11) and (14) there exists $r_0 \ge r_2$ such that

$$(\log r)\log\left(\frac{\log(c_0 r)}{\log r}\right) > \pi\sqrt{C_2/C_1} \quad \forall r \ge r_0. \tag{15}$$

By (13) and (15), we get (2) and (5). By (12) and a direct computation (3) and (4) holds with $b_1 = \sqrt{C_1}/(\sqrt{2}c_0 C_2)$, $b_2 = \sqrt{C_1}\log 2$, and the proposition follows. $\square$

Proof of Corollary 3: By (10) and the results of [ERV] there exists a constant $C_2 > 0$ such that

$$u(x,t) \le \frac{C_2}{|x|^2(\log|x|)^2} \quad \forall |x| > 1, 0 < t < T \tag{16}$$

and for any $t_0 \in (0,T)$ there exists a constant $r_1 > 1$ such that

$$u(x,t) \ge \frac{(3/2)t}{|x|^2(\log|x|)^2} \quad \forall |x| \ge r_1, 0 < t \le t_0. \tag{17}$$

By (16), (17), Theorem 1 and Proposition 2, the corollary follows. $\square$

We will now assume that g is a metric on $\mathbb{R}^2$ with finite total area that satisfies (1), (2), (3), (4) and (5) for some constants $r_0 > 1$, $c_0 > 1$, $b_1 > 0$, $b_2 > 0$, $\delta > 0$ where $\lambda_1(r)$, $\lambda_2(r)$, are positive monotone decreasing functions on $[r_0,\infty)$ for the rest of the paper. Let $b_0 = \min(b_1, 4b_2/A)$. Suppose (9) holds. Let $\{\gamma_k\}_{k=1}^\infty$ be a sequence of closed simple curves on $\mathbb{R}^2$ such that

$$I(\gamma_k) \to I \quad \text{as } k\to\infty \quad \text{and} \quad I(\gamma_k) < b_0 \quad \forall k \in \mathbb{Z}^+. \tag{18}$$

We will show that the sequence $\{\gamma_k\}_{k=1}^\infty$ is contained in some compact set of $\mathbb{R}^2$. Let Ω_k be the region inside γ_k and $r_k = \min_{x\in\gamma_k}|x|$. Let $L_e(\gamma_k)$ be the length of γ_k and $|\Omega_k|$ be the area of Ω_k with respect to the Euclidean metric. We choose $r_0' > r_0$ such that

$$\mathrm{Vol}_g(\mathbb{R}^2 \setminus B_{r_0'}) \le \frac{A}{4} \quad \forall k \in \mathbb{Z}^+. \tag{19}$$

Lemma 4. *The sequence r_k is uniformly bounded.*

Proof: Suppose the lemma is not true. Then there exists a subsequence of r_k which we may assume without loss of generality to be the sequence itself such that

$$r_k > r_0' \quad \forall k \in \mathbb{Z}^+ \tag{20}$$

and $r_k \to \infty$ as $k \to \infty$. Let $\widetilde{\gamma}_k = \partial B_{r_k}$. We choose a point $x_k \in \gamma_k \cap \partial B_{r_k}$ and let $\gamma_k : [0, 2\pi] \to \mathbb{R}^2$ be a parametrization of the curve γ_k such that $x_k = \gamma_k(0) = \gamma_k(2\pi)$. Since for any $k \in \mathbb{Z}^+$ either $0 \in \Omega_k$ or $0 \in \mathbb{R}^2 \setminus \Omega_k$ holds, thus either

$$0 \in \Omega_k \quad \text{ for infinitely many k} \tag{21}$$

or

$$0 \in \mathbb{R}^2 \setminus \Omega_k \quad \text{ for infinitely many k} \tag{22}$$

holds. We need the following result for the proof of the lemma.
<u>Claim 1</u>: There exists only finitely many k such that $\gamma_k \cap (\mathbb{R}^2 \setminus \overline{B}_{c_0 r_k}) \neq \emptyset$.
Proof of Claim 1: Suppose claim 1 is false. Then there exists infinitely many k such that $\gamma_k \cap (\mathbb{R}^2 \setminus \overline{B}_{c_0 r_k}) \neq \emptyset$. Without loss of generality we may assume that

$$\gamma_k \cap (\mathbb{R}^2 \setminus \overline{B}_{c_0 r_k}) \neq \emptyset \quad \forall k \in \mathbb{Z}^+. \tag{23}$$

By (23) there exists $\phi_0 \in (0, 2\pi)$ such that

$$|\gamma_k(\phi_0)| > c_0 r_k.$$

Hence there exists $0 < \phi_1 < \phi_0 < \phi_2 < 2\pi$ such that

$$\gamma_k(\phi_1) = \gamma_k(\phi_2) = c_0 r_k$$

and

$$r_k \le |\gamma_k(\phi)| \le c_0 r_k \quad \forall \phi \in (0, \phi_1) \cup (\phi_2, 2\pi).$$

Then by (1),

$$\begin{aligned}
L(\gamma_k) =& \int_0^{2\pi} (g_{ij}\dot{\gamma}_k^i \dot{\gamma}_k^j)^{\frac{1}{2}}\, d\phi \\
\ge& \left(\int_0^{\phi_1} + \int_{\phi_2}^{2\pi} \right) (g_{ij}\dot{\gamma}_k^i \dot{\gamma}_k^j)^{\frac{1}{2}}\, d\phi \\
\ge& \left(\int_0^{\phi_1} + \int_{\phi_2}^{2\pi} \right) \sqrt{\lambda_1(r)} \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2 \left(\frac{d\theta}{d\phi}\right)^2}\, d\phi \\
\ge& 2 \int_{r_k}^{c_0 r_k} \sqrt{\lambda_1(r)}\, dr
\end{aligned} \tag{24}$$

and

$$2\pi r_k \sqrt{\lambda_1(r_k)} \le L(\widetilde{\gamma}_k) = \int_0^{2\pi} (g_{ij}\dot{\widetilde{\gamma}}_k^i \dot{\widetilde{\gamma}}_k^j)^{\frac{1}{2}}\, d\phi \le 2\pi r_k \sqrt{\lambda_2(r_k)}. \tag{25}$$

By (2), (24) and (25),

$$L(\widetilde{\gamma}_k) \le L(\gamma_k). \tag{26}$$

Suppose (21) holds. Without loss of generality we may assume that $0 \in \Omega_k$ for all $k \in \mathbb{Z}^+$. Then $B_{r_k} \subset \Omega_k$ for all $k \in \mathbb{Z}^+$. Hence by (19), (20),

$$A_{out}(\gamma_k) \le \mathrm{Vol}_g(\mathbb{R}^2 \setminus B_{r_k}) \le \frac{A}{4} \quad \forall k \in \mathbb{Z}^+ \tag{27}$$

and

$$\frac{3A}{4} \le \mathrm{Vol}_g(B_{r_k}) \le A_{in}(\gamma_k) \le A \quad \forall k \in \mathbb{Z}^+. \tag{28}$$

We will now show that the circle $\widetilde{\gamma}_k = \partial B_{r_k}$ satisfies

$$I(\widetilde{\gamma}_k) \le I(\gamma_k). \tag{29}$$

Let $\varepsilon = A_{out}(\widetilde{\gamma}_k) - A_{out}(\gamma_k)$. Then $\varepsilon = A_{in}(\gamma_k) - A_{in}(\widetilde{\gamma}_k)$. Since $\widetilde{\gamma}_k \subset \overline{\Omega}_k$ and the region between γ_k and $\widetilde{\gamma}_k$ is contained in $\mathbb{R}^2 \setminus B_{r_k}$, by (27),

$$0 \le \varepsilon \le \frac{A}{4}. \tag{30}$$

Hence by (27) and (30),

$$\begin{aligned}\frac{1}{A_{in}(\widetilde{\gamma}_k)} + \frac{1}{A_{out}(\widetilde{\gamma}_k)} &= \frac{A}{A_{in}(\widetilde{\gamma}_k)A_{out}(\widetilde{\gamma}_k)} = \frac{A}{(A_{in}(\gamma_k) - \varepsilon)(A_{out}(\gamma_k) + \varepsilon)} \\ &\le \frac{A}{A_{in}(\gamma_k)A_{out}(\gamma_k)} = \frac{1}{A_{in}(\gamma_k)} + \frac{1}{A_{out}(\gamma_k)}.\end{aligned} \tag{31}$$

By (26) and (31) we get (29). Now by (1),

$$A_{out}(\widetilde{\gamma}_k) = \int_{\mathbb{R}^2 \setminus B_{r_k}} \sqrt{\det g_{ij}}\, dx \le 2\pi \int_{r_k}^{\infty} \rho \lambda_2(\rho)\, d\rho. \tag{32}$$

By (3), (25), (29) and (32),

$$I(\gamma_k) \ge \frac{L(\widetilde{\gamma}_k)}{A_{out}(\widetilde{\gamma}_k)} + \frac{L(\widetilde{\gamma}_k)}{A_{in}(\widetilde{\gamma}_k)} \ge b_1. \tag{33}$$

Letting $k \to \infty$ in (33),

$$I \ge b_1. \tag{34}$$

This contradicts (9) and the definition of b_0. Hence (21) does not hold.

Suppose (22) holds. Without loss of generality we may assume that $0 \in \mathbb{R}^2 \setminus \Omega_k$ for all $k \in \mathbb{Z}^+$. Then by (20) $0 \in \mathbb{R}^2 \setminus \overline{\Omega}_k$ and $B_{r_k} \subset \mathbb{R}^2 \setminus \overline{\Omega}_k$ for any $k \in \mathbb{Z}^+$. By

an argument similar to the proof of (27) and (28) but with the role of $A_{in}(\gamma_k)$ and $A_{out}(\gamma_k)$ being interchanged in the proof we get

$$\begin{cases} A_{in}(\gamma_k) \le \mathrm{Vol}_g(\mathbb{R}^2 \setminus B_{r_k}) \le \dfrac{A}{4} & \forall k \in \mathbb{Z}^+ \\ \dfrac{3A}{4} \le A_{out}(\gamma_k) \le A & \forall k \in \mathbb{Z}^+. \end{cases} \tag{35}$$

Similarly by interchanging the role of $A_{in}(\gamma_k)$ and $A_{out}(\gamma_k)$ and replacing ε by $\varepsilon' = A_{out}(\widetilde{\gamma}_k) - A_{in}(\gamma_k) = A_{out}(\gamma_k) - A_{in}(\widetilde{\gamma}_k)$ in the proof of (29)–(33) above, we get that $0 \le \varepsilon' \le A/4$ and (29), (33), still holds. Letting $k \to \infty$ in (33), we get (34). This again contradicts (9) and the definition of b_0. Thus (22) does not hold and claim 1 follows.

We will now continue with the proof of the lemma. By claim 1 there exists $k_0 \in \mathbb{Z}^+$ such that

$$\begin{aligned} & \gamma_k \cap (\mathbb{R}^2 \setminus \overline{B}_{c_0 r_k}) = \emptyset \quad \forall k \ge k_0 \\ \Rightarrow \quad & \gamma_k \subset \overline{B}_{c_0 r_k} \setminus B_{r_k} \quad \forall k \ge k_0. \end{aligned} \tag{36}$$

Note that either (21) or (22) holds. Suppose (21) holds. Without loss of generality we may assume that $0 \in \Omega_k$ for all $k \ge k_0$. Then $B_{r_k} \subset \Omega_k$ for all $k \ge k_0$. Hence by (1) and (36),

$$\begin{aligned} L(\gamma_k) = & \int_0^{2\pi} (g_{ij}\dot{\gamma}_k^i \dot{\gamma}_k^j)^{\frac{1}{2}}\, d\phi \\ \ge & \sqrt{\lambda_1(c_0 r_k)} \int_0^{2\pi} \left(\left(\frac{dr}{d\phi}\right)^2 + r^2 \left(\frac{d\theta}{d\phi}\right)^2 \right)^{\frac{1}{2}} d\phi \\ \ge & 2\pi r_k \sqrt{\lambda_1(c_0 r_k)} \quad \forall k \ge k_0 \end{aligned} \tag{37}$$

and

$$A_{out}(\gamma_k) \le \int_{\mathbb{R}^2 \setminus B_{r_k}} \sqrt{\det g_{ij}}\, dx \le 2\pi \int_{r_k}^\infty \rho \lambda_2(\rho)\, d\rho \quad \forall k \ge k_0. \tag{38}$$

By (3), (37) and (38),

$$I(\gamma_k) \ge \frac{L(\gamma_k)}{A_{out}(\gamma_k)} \ge \frac{r_k \sqrt{\lambda_1(c_0 r_k)}}{\int_{r_k}^\infty \rho \lambda_2(\rho)\, d\rho} \ge b_1 \quad \forall k \ge k_0. \tag{39}$$

Letting $k \to \infty$ in (39), we get (34). Since (34) contradicts (9) and the definition of b_0, (21) does not hold. Hence (22) holds. By (20) and (22) we may assume without loss of generality that $0 \in \mathbb{R}^2 \setminus \overline{\Omega}_k$ for all $k \ge k_0$. Then $B_{r_k} \subset \mathbb{R}^2 \setminus \overline{\Omega}_k$ for all $k \ge k_0$. Hence Ω_k is contractible to a point in $\overline{B}_{c_0 r_k} \setminus B_{r_k}$ for all $k \ge k_0$. By (1),

$$L(\gamma_k) = \int_0^{2\pi} (g_{ij}\dot{\gamma}_k^i \dot{\gamma}_k^j)^{\frac{1}{2}}\, d\phi \ge \sqrt{\lambda_1(c_0 r_k)} L_e(\gamma_k) \quad \forall k \ge k_0. \tag{40}$$

By the isoperimetric inequality,

$$4\pi|\Omega_k| \le L_e(\gamma_k)^2. \tag{41}$$

Then by (40) and (41),

$$L(\gamma_k) \ge 2(\pi\lambda_1(c_0 r_k)|\Omega_k|)^{\frac{1}{2}} \quad \forall k \ge k_0. \tag{42}$$

Now

$$A_{in}(\gamma_k) = \int_{\Omega_k} \sqrt{\text{det} g_{ij}}\, dx \le \lambda_2(r_k)|\Omega_k| \quad \forall k \ge k_0. \tag{43}$$

By (5), (42) and (43),

$$\begin{aligned} & L(\gamma_k) \ge 2\pi^{\frac{1}{2}} \left(\frac{\lambda_1(c_0 r_k)}{\lambda_2(r_k)}\right)^{\frac{1}{2}} A_{in}(\gamma_k)^{\frac{1}{2}} \ge 2(\pi\delta)^{\frac{1}{2}} A_{in}(\gamma_k)^{\frac{1}{2}} \quad \forall k \ge k_0 \\ \Rightarrow \quad & I(\gamma_k) \ge \frac{L(\gamma_k)}{A_{in}(\gamma_k)} \ge 2(\pi\delta)^{\frac{1}{2}} A_{in}(\gamma_k)^{-\frac{1}{2}} \quad \forall k \ge k_0. \end{aligned} \tag{44}$$

Since $\Omega_k \subset \mathbb{R}^2 \setminus B_{r_k}$,

$$A_{in}(\gamma_k) \to 0 \qquad \text{as } k \to \infty. \tag{45}$$

Letting $k \to \infty$ in (44) by (45) we get $I = \infty$. This contradicts (9). Hence (22) does not hold and the lemma follows. $\square$

By Lemma 4 there exists a constant $a_1 > r_0$ such that

$$r_k \le a_1 \quad \forall k \in \mathbb{Z}^+. \tag{46}$$

Lemma 5. $\gamma_k \in \overline{B}_{a_1^2} \quad \forall k \in \mathbb{Z}^+$.

Proof: Let $\rho_k = \max_{\gamma_k} |x|$. Suppose the lemma does not hold. Then there exists a subsequence of ρ_k which we may assume without loss of generality to be the sequence itself such that

$$\rho_k > a_1^2 \quad \forall k \in \mathbb{Z}^+. \tag{47}$$

By (1), (4), (46), (47) and an argument similar to the proof of (24),

$$L(\gamma_k) \ge \int_{a_1}^{a_1^2} \sqrt{\lambda_1(\rho)}\, d\rho \ge b_2 \quad \forall k \in \mathbb{Z}^+. \tag{48}$$

Hence by (48),

$$\begin{aligned} & I(\gamma_k) = \frac{AL(\gamma_k)}{A_{in}(\gamma_k)A_{out}(\gamma_k)} \ge \frac{Ab_2}{(A/2)^2} = \frac{4b_2}{A} \quad \forall k \in \mathbb{Z}^+ \\ \Rightarrow \quad & I \ge \frac{4b_2}{A} \qquad \text{as } k \to \infty. \end{aligned}$$

This contradicts (9) and the definition of b_0. Hence the lemma follows. $\square$

Let $L_k = L(\gamma_k)$. Since $\overline{B}_{a_1^2}$ is compact, there exists constants $c_2 > c_1 > 0$ such that

$$c_1\delta_{ij} \le g_{ij} \le c_2\delta_{ij} \quad \text{ on } \overline{B}_{a_1^2}. \tag{49}$$

Lemma 6. *There exists a constant $\delta_1 > 0$ such that $L_k \ge \delta_1 \quad \forall k \in \mathbb{Z}^+$.*

Proof: By (49),

$$\begin{cases} c_1^{\frac{1}{2}} L_e(\gamma_k) \le L_k \le c_2^{\frac{1}{2}} L_e(\gamma_k) \quad \forall k \in \mathbb{Z}^+ \\ c_1|\Omega_k| \le A_{in}(\gamma_k) \le c_2|\Omega_k| \quad \forall k \in \mathbb{Z}^+. \end{cases} \tag{50}$$

By (18), (41) and (50),

$$b_0 > \frac{L_k}{A_{in}(\gamma_k)} \ge \frac{c_1^{\frac{1}{2}} L_e(\gamma_k)}{c_2|\Omega_k|} \ge \frac{c_1^{\frac{1}{2}}}{c_2} \cdot \frac{L_e(\gamma_k)}{(L_e(\gamma_k)^2/4\pi)} \ge \frac{4\pi c_1^{\frac{1}{2}}}{c_2 L_e(\gamma_k)} \quad \forall k \in \mathbb{Z}^+$$
$$\Rightarrow \quad L_k \ge c_1^{\frac{1}{2}} L_e(\gamma_k) \ge \frac{4\pi c_1}{c_2 b_0} \quad \forall k \in \mathbb{Z}^+$$

and the lemma follows. $\square$

By the proof of Lemma 6 we have the following corollary.

Corollary 7. *For any constant $C_1 > 0$ there exists a constant $\delta_1 > 0$ such that*

$$L(\gamma) > \delta_1$$

for any simple closed curve $\gamma \subset \overline{B}_{a_1^2}$ satisfying

$$I(\gamma) < C_1. \tag{51}$$

By (6) and Corollary 7 we have the following corollary.

Corollary 8. *For any constant $C_1 > 0$ there exists a constant $\delta_2 > 0$ such that*

$$A_{in}(\gamma) > \delta_2 \quad \text{ and } \quad A_{out}(\gamma) > \delta_2$$

for any simple closed curve $\gamma \subset \overline{B}_{a_1^2}$ satisfying (51).

Lemma 9. *There exists a constant $C_2 > 0$ such that the following holds. Suppose $\beta \subset \overline{B}_{a_1^2}$ is a closed simple curve. Then under the curve shrinking flow*

$$\frac{\partial \beta}{\partial \tau}(s, \tau) = k\vec{N} \tag{52}$$

with $\beta(s, 0) = \beta(s)$ where for each $\tau \ge 0$, $k(\cdot, \tau)$ is the curvature, $\vec{N}$ is the unit inner normal, and s is the arc length of the curve $\beta(\cdot, \tau)$ with respect to the metric g, there exists $\tau_0 \ge 0$ such that the curve $\beta^{\tau_0} = \beta(\cdot, \tau_0) \subset \overline{B}_{a_1^2}$ satisfies $I(\beta^{\tau_0}) \le I(\beta)$ and

$$\int k(s, \tau_0)^2 \, ds \le C_2.$$

Proof: Since the proof is similar to the proof of [DH] and the Lemma on P.197 of [H2], we will only sketch the proof here. Let $\beta^\tau = \beta(\cdot,\tau)$ and write

$$L(\tau) = L_g(\beta(\cdot,\tau)),\ I(\tau) = I(\beta^\tau) = I_g(\beta(\cdot,\tau)),$$

and the areas

$$A_{in}(\tau) = A_{in}(\beta(\cdot,\tau)),\ A_{out}(\tau) = A_{out}(\beta(\cdot,\tau)),$$

with respect to the metric g. Let $T_1 > 0$ be the maximal existence time of the solution of (52). Then

$$\beta^\tau \subset \overline{B}_{a_1^2} \quad \forall 0 \le \tau < T_1. \tag{53}$$

Similar to the result on P.196 of [H2] we have

$$\frac{\partial A_{in}}{\partial \tau} = -\int k\,ds, \quad \frac{\partial A_{out}}{\partial \tau} = \int k\,ds, \quad \frac{\partial L}{\partial \tau} = -\int k^2\,ds \tag{54}$$

and

$$\int k\,ds + \int_{\Omega(\tau)} K\,dV_g = 2\pi \tag{55}$$

by the Gauss-Bonnet theorem where K is the Gauss curvature with respect to g and $\Omega(\tau) \subset \overline{B}_{a_1^2}$ is the region enclosed by the curve $\beta(s,\tau)$. Let $C_1 = 2I(\beta)$. By continuity there exists a constant $0 < \delta_0 < T_1$ such that

$$I(\tau) < C_1 \quad \forall 0 \le \tau \le \delta_0. \tag{56}$$

By (56), Corollary 7, and Corollary 8 there exist constants $\delta_1 > 0$, $\delta_2 > 0$, such that

$$L(\tau) > \delta_1, \quad A_{in}(\tau) > \delta_2, \quad A_{out}(\tau) > \delta_2 \quad \forall 0 \le \tau \le \delta_0. \tag{57}$$

Now

$$\frac{\partial}{\partial \tau}(\log I(\tau)) = \frac{1}{L}\frac{\partial L}{\partial \tau} - \frac{1}{A_{in}}\frac{\partial A_{in}}{\partial \tau} - \frac{1}{A_{out}}\frac{\partial A_{out}}{\partial \tau} + \frac{1}{A}\frac{\partial A}{\partial \tau}. \tag{58}$$

By (53) and (55) $\int k\,ds$ is uniformly bounded for all $0 \le \tau < T_1$. Then by (54), (55), (57), and (58), there exists a constant $C_2 > 0$ independent of δ_0 such that

$$\frac{\partial}{\partial \tau}(\log I(\tau)) < 0$$

for any $\tau \in (0, \delta_0]$ satisfying

$$\int k(s,\tau)^2\,ds > C_2.$$

If

$$\int k(s,0)^2\,ds \le C_2,$$

we set $\tau_0 = 0$ and we are done. If

$$\int k(s,0)^2\,ds > C_2,$$

then either there exists $\tau_0 \in (0, \delta_0]$ such that

$$\int k(s,\tau_0)^2\,ds = C_2 \quad \text{and} \quad \int k(s,\tau)^2\,ds > C_2 \quad \forall 0 \le \tau < \tau_0 \tag{59}$$

or

$$\int k(s,\tau)^2\,ds > C_2 \quad \forall 0 \le \tau \le \delta_0. \tag{60}$$

If (59) holds, since $I(\tau_0) \le I(0)$ we are done. If (60) holds, since $I(\delta_0) \le I(0)$ we can repeat the above the argument a finite number of times. Then either

(a) there exists $\tau_0 \in (0, T_1)$ such that (59) holds

or

(b)

$$\int k(s,\tau)^2\,ds > C_2 \quad \forall 0 \le \tau < T_1 \tag{61}$$

holds.

If (b) holds, then similar to the proof of the Lemma on P.197 of [H2] by (57) we get a contradiction to the Grayson theorem ([H2],[Gr1],[Gr2]) for curve shortening flow. Hence (a) holds. Since $I(\tau_0) \le I(0)$, the lemma follows. □

To complete the proof of Theorem 1 we also need the following technical lemma.

Lemma 10. *For any positive numbers $\alpha_1, \alpha_2, A_1, A_2, A_3$ we have*

$$(\alpha_1 + \alpha_2)\left(\frac{1}{A_2} + \frac{1}{A_1 + A_3}\right) \ge \min\left\{\alpha_1\left(\frac{1}{A_1} + \frac{1}{A_2 + A_3}\right), \alpha_2\left(\frac{1}{A_3} + \frac{1}{A_1 + A_2}\right)\right\}. \tag{62}$$

Proof: Suppose (62) does not hold. Then

$$\begin{aligned} &(\alpha_1 + \alpha_2)\left(\frac{1}{A_2} + \frac{1}{A_1 + A_3}\right) \le \alpha_1\left(\frac{1}{A_1} + \frac{1}{A_2 + A_3}\right) \\ \Rightarrow\quad &\frac{A_1(A_2 + A_3)}{A_2(A_1 + A_3)} \le \frac{\alpha_1}{\alpha_1 + \alpha_2} \end{aligned} \tag{63}$$

and

$$\begin{aligned} &(\alpha_1 + \alpha_2)\left(\frac{1}{A_2} + \frac{1}{A_1 + A_3}\right) \le \alpha_2\left(\frac{1}{A_3} + \frac{1}{A_1 + A_2}\right) \\ \Rightarrow\quad &\frac{A_3(A_1 + A_2)}{A_2(A_1 + A_3)} \le \frac{\alpha_2}{\alpha_1 + \alpha_2}. \end{aligned} \tag{64}$$

Summing (63) and (64),

$$\frac{2A_1A_3}{A_2(A_1+A_3)} \le 0 \quad \Rightarrow \quad A_1 = 0 \text{ or } A_3 = 0.$$

Contradiction arises. Hence (62) holds and the lemma follows. □

We are now ready for the proof of Theorem 1.

Proof of Theorem 1: Since the proof is similar to the proof of [H1] and [H2] we will only sketch the argument here. Let $C_2 > 0$ be given by Lemma 9 and $\delta_1 > 0$ be given by Corollary 7 with $C_1 = b_0$. By Lemma 5, Lemma 6, Corollary 7, Lemma 9 and an argument similar to the proof of [H2] for each $j \in \mathbb{Z}^+$ there exists a closed simple curve $\overline{\gamma}_j \subset \overline{B}_{a_1^2}$ satisfying

$$I(\overline{\gamma}_j) \le I(\gamma_j) \quad \text{and} \quad L(\overline{\gamma}_j) \ge \delta_1 \quad \forall j \in \mathbb{Z}^+$$

and

$$\int_{\overline{\gamma}_j} k^2\, ds \le C_2 \tag{65}$$

where k is the curvature of $\overline{\gamma}_j$. By (65) and the same argument as that on P. 197-199 of [H2] $\overline{\gamma}_j$ are locally uniformly bounded in L_2^1 and $C^{1+\frac{1}{2}}$. Hence $\overline{\gamma}_j$ has a sequence which we may assume without loss of generality to be the sequence itself that converges uniformly in L_p^1 for any $1 < p < 2$ and in $C^{1+\alpha}$ for any $0 < \alpha < 1/2$ as $j \to \infty$ to some closed immersed curve $\gamma \subset \overline{B}_{a_1^2}$. Moreover γ satisfies

$$I = I(\gamma) \quad \text{and} \quad L(\gamma) \ge \delta_1.$$

Since γ is the limit of embedded curves, γ cannot cross itself and at worst it will be self tangent. Suppose γ is self tangent. Without loss of generality we may assume that γ is only self tangent at one point. Then $\gamma = \beta_1 \cup \beta_2$ with $\beta_1 \cap \beta_2$ being a single point where β_1, β_2, are simple closed curves. Then $A_{in}(\gamma) = A_{in}(\beta_1) + A_{in}(\beta_2)$, $A_{out}(\beta_1) = A_{out}(\gamma) + A_{in}(\beta_2)$, $A_{out}(\beta_2) = A_{out}(\gamma) + A_{in}(\beta_1)$, and $L(\gamma) = L(\beta_1) + L(\beta_2)$. Let $L_1 = L(\beta_1)$ and $L_2 = L(\beta_2)$. By Lemma 10,

$$\begin{aligned}
&(L_1 + L_2)\left(\frac{1}{A_{out}(\gamma)} + \frac{1}{A_{in}(\beta_1) + A_{in}(\beta_2)}\right)\\
&\ge \min\left\{L_1\left(\frac{1}{A_{in}(\beta_1)} + \frac{1}{A_{out}(\gamma) + A_{in}(\beta_2)}\right), L_2\left(\frac{1}{A_{in}(\beta_2)} + \frac{1}{A_{out}(\gamma) + A_{in}(\beta_1)}\right)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
& L(\gamma)\left(\frac{1}{A_{out}(\gamma)} + \frac{1}{A_{in}(\gamma)}\right)\\
&\qquad \ge \min\left\{L_1\left(\frac{1}{A_{in}(\beta_1)} + \frac{1}{A_{out}(\beta_1)}\right), L_2\left(\frac{1}{A_{in}(\beta_2)} + \frac{1}{A_{out}(\beta_2)}\right)\right\}\\
\Rightarrow \quad & I(\gamma) \ge \min(I(\beta_1), I(\beta_2))\\
\Rightarrow \quad & I(\gamma) = \min(I(\beta_1), I(\beta_2)).
\end{aligned}$$

Without loss of generality we may assume that $I(\gamma) = I(\beta_1)$. Then β_1 is a simple closed curve which attains the minimum. Similar to the proof of [H2], by a variation argument β_1 has constant curvature

$$k = L\left(\frac{1}{A_{in}} - \frac{1}{A_{out}}\right).$$

Hence β_1 is smooth and the theorem follows.

□

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